Taylor's series (17/07)

Solution of ordinary differential Equations

Single Step by Taylor's Series method:

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y),\ f(x_0) = y_0$$

If $y(x)$ is the solution, then

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots\cdots$$

W.K.T $h = x - x_0$

$$y(x) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Prob:1 Find the taylor's series method the values of y at $x = 0.1$ and $x = 0.2$, correct to four decimal places from $\frac{dy}{dx} = x^2 y - 1$, $y(0) = 1$.

Soln: Here $x_0 = 0$, $y_0 = 1$

$y' = x^2 y - 1$, $y_0' = 0 - 1 = -1$

$y'' = 2xy + x^2 y'$,

$y_0'' = 2x_0 y_0 + x_0^2 y_0'$

$= 2(0)(1) + (0)(-1) = 0$

$y''' = 2(xy' + y) + x^2 y'' + 2xy'$

$= 4xy' + 2y + x^2 y''$.

$y_0''' = 4x_0 y_0' + 2y_0 + x_0^2 y_0''$

$= 4(0) + 2(1) + 0 = 2$

$y^{iv} = 4[xy'' + y'] + 2y' + x^2 y''' + 2xy''$

$= 6y' + 6xy'' + x^2 y'''$

$y_0^{iv} = 6y_0' + 6x_0 y_0'' + x_0^2 y_0''' = -6$
$6(-1) + 0 + 0$

Taylor's series of $y(x)$ about $x_0 = 0$ is given by,

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

Here $x_0 = 0$

$$y(x) = y_0 + \frac{x}{1!} y_0' + \frac{x^2}{2!} y_0'' + \cdots$$

$$= 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$$

$$y(x) = 1 - x + \frac{2x^3}{6} - \frac{6x^4}{24} + \cdots$$

$$y(0.1) = 1 - 0.1 + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

$$= 1 - 0.1 + 0.0003 - 0.00025$$

$$\boxed{y(0.1) = 0.90005}$$

$$y(0.2) = 1 - 0.2 + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \cdots$$

$$= 1 - 0.2 + 0.0026 - 0.0004$$

$$\boxed{y(0.2) = 0.8022}$$

2. By Taylor's series find $y(0.1)$ & $y(0.2)$ given that $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ correct to four decimal places.

Soln: Given $y' = x^2 + y^2$, $y(0) = 1$

$x_0 = 0$ & $y_0 = 1$.

$y' = x^2 + y^2$, $y_0' = x_0^2 + y_0^2 = 0 + 1 = 1$

$y'' = 2x + 2yy'$

$y_0'' = 2x_0 + 2y_0 y_0' = 2(0) + 2(1)(1)$

$y_0'' = 2$

$y''' = 2 + 2yy'' + 2y'y'$

$= 2 + 2yy'' + 2(y')^2$

$y_0''' = 2 + 2y_0 y_0'' + 2(y_0')^2$

$= 2 + 2(1)(2) + 2(1)^2$

$= 2 + 4 + 2 = 8$

$y^{iv} = 2y'y'' + 2yy''' + 4y'(y'')$

$= 2yy''' + 6y'y''$

$$y_0^{iv} = 2 y_0 y_0''' + 6 y_0' y_0''$$

$$= 2(1)(8) + 6(1)(2)$$

$$= 28$$

W.K.T

$$y = y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0''$$

$$+ \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

Here $x_0 = 0$

$$y = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(2) + \frac{x^3}{3!}(8)$$

$$+ \frac{x^4}{4!}(28) + \cdots$$

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(8)$$

$$+ \frac{(0.1)^4}{24}(28) + \cdots$$

$$= 1 + 0.1 + 0.01 + 0.0013 + 0.0001$$

$$\boxed{y(0.1) = 1.1114}$$

$$\ldots + 0.2 + (0.2)^2 (2)$$

$$y(0.2) = 1 + \ldots \frac{\quad}{2}$$

$$+ \frac{(0.2)^3}{6}(8) + \frac{(0.2)^4}{24}(28) + \ldots$$

$$= 1 + 0.2 + 0.04 + 0.01066$$

$$+ 0.00186$$

$$\boxed{y(0.2) = 1.2525}$$

3. Evaluate $y(0.1)$ and $y(0.2)$ Correct to four decimal places by Taylor's series method, if $y(x)$ satisfies $y' = xy + 1$, $y(0) = 1$.

Soln: Given $y' = xy + 1$, $y(0) = 1$

Here $x_0 = 0$, $y_0 = 1$

$$y' = xy + 1 \qquad y_0' = x_0 y_0 + 1 = 1$$

$$y'' = xy' + y$$

$$y_0'' = x_0 y_0' + y_0 = 0 + 1 = 1$$

$$y''' = xy'' + y' + y' = xy'' + 2y'$$

$$y_0''' = x_0 y_0'' + 2y_0' = 0 + 2 \times 1 = 2$$

$$y^{IV} = xy''' + y'' + 2y'' = xy''' + 3y''$$

$$y_0^{IV} = x_0 y_0''' + 3y_0'' = 3(1) = 3$$

W.K.T

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

$$y(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(2)$$

$$+\frac{x^4}{4!}(3)+\cdots$$

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{\cancel{6}\,3}(\cancel{2})$$

$$+\frac{(0.1)^4}{\cancel{24}\,8}(\cancel{3}) + \cdots$$

$$= 1 + 0.1 + 0.005 + 0.0003$$
$$+ 0.00001$$

$$\boxed{y(0.1) = 1.1053}$$

$$y(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3}$$

$$+\frac{(0.2)^4}{8} + \cdots$$

$$= 1 + 0.2 + 0.02 + 0.0026$$
$$+ 0.0002$$

$$\boxed{y(0.2) = 1.2228}$$